Hexagonal Wilson Loops in Planar $\mathcal{N} = 4$ SYM Theory at Finite Coupling

Benjamin Basso$^{\lozenge}$, Amit Sever$^{\square}$ and Pedro Vieira$^{\lozenge}$

$^{\lozenge}$*Laboratoire de Physique Théorique, École Normale Supérieure, Paris 75005, France*
$^{\square}$*School of Physics and Astronomy, Tel Aviv University, Ramat Aviv 69978, Israel*
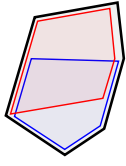$^{\lozenge}$*Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada*

We report on the complete OPE series for the 6-gluon MHV and NMHV amplitudes in planar $\mathcal{N} = 4$ SYM theory. Namely, we provide a finite coupling prediction for all the terms in the expansion of these amplitudes around the collinear limit. These furnish a non-perturbative representation of the full amplitudes.

In [1], building on [2], the so-called POPE (Pentagon Operator Product Expansion) program was put forward. Its goal is to put to full use the integrability of the chromodynamic flux tube of planar $\mathcal{N} = 4$ SYM theory to study scattering amplitudes and polygonal Wilson loops at any value of the 't Hooft coupling $g = \sqrt{\lambda}/4\pi$.

Since then, through a series of works [3–12], a simple physical and mathematical structure underpinning these objects has emerged. In this short letter, we collect all known information about hexagonal loops – i.e. 6-gluon scattering amplitudes – and present their complete non-perturbative expression as a POPE expansion.

Within the POPE formalism, the renormalized hexagonal Wilson loops [1, 2] are given by an infinite sum

$$\mathcal{W}_6 = \quad = \sum_n \frac{1}{S_n} \int \frac{du_1 \dots du_n}{(2\pi)^n} \Pi(\{u_i\}) \tag{1}$$

over all possible n-particle states of the colour flux tube, integrated over their n-dimensional phase space, and with S_n a symmetry factor for identical particles; it is equal to $n_1! n_2! \dots n_k!$ with $n = n_1 + \dots n_k$ and n_j the number of identical particles of a given kind. The POPE integrand in (1) stands for the probability that the corresponding state gets successively produced and annihilated on, respectively, the bottom and top pentagons inside the hexagon. It can be conveniently decomposed as a product of three simpler building blocks as

$$\Pi = \Pi_{\text{dyn}} \times \Pi_{\text{FF}} \times \Pi_{\text{mat}} . \tag{2}$$

The product of the first two terms was spelled out explicitly in [12]. The last factor, the so-called matrix part, deals with the $SU(4)$ R-symmetry degrees of freedom of the excitations, and was partially unveiled in [6] for scalar excitations in the hexagon MHV Wilson loop. Below we quote this last missing factor for any state on any hexagon. The derivation of this matrix part factor and its generalization to any polygon will be given in [13].

The *dynamical part* neatly factorizes as

$$\Pi_{\text{dyn}} = \prod_i \mu(u_i) \, e^{-E(u_i)\tau + ip(u_i)\sigma + im_i\phi} \prod_{i<j} \frac{1}{|P(u_i|u_j)|^2} ,$$

where τ, σ and ϕ are the thee independent conformal cross ratios we denote in the OPE as time, space and angle, see [1]. The hexagon geometry enters only in these exponentials. Most importantly, we have the pentagon transition link $P(u_i|u_j)$ between excitation i and excitation j and their corresponding flux tube measures $\mu(u_i)$.

Next we have the *form factor part* given once more by a simple factorized product

$$\Pi_{\text{FF}} = g^{\frac{r_b(r_b-4)}{8} + \frac{r_t(r_t-4)}{8}} \times \prod_i h(u_i)^{r_t - r_b} ,$$

where r_b $(r_t) = 0, 1, \dots, 4$ is the R-charge carried by the bottom (top) pentagon [10, 12]. For NMHV hexagons we have $r_t = 4 - r_b$, while for MHV and N²MHV amplitudes we have $r_b = r_t = 0$ and $r_b = r_t = 4$, respectively. Therefore this factor is only needed to describe (any of the five independent) NMHV hexagons.

Note that P, μ, h and the propagation charges implicitly depend on the flavour of the excitation; e.g. if excitation i is a fermion ψ and excitation j a scalar ϕ, then in the notation of [10] we have $P(u_i|u_j) = P_{\psi\phi}(u_i|u_j)$, $\mu(u_i) = \mu_\psi(u_i)$, $h(u_i) = h_\psi(u_i)$, $E(u_i) = E_\psi(u_i)$ etc. All form factors, transitions and measures are summarized in [12] while the dispersion relations can be found in [5]. For scalar and gluon excitations the contour of the integration in (1) is over the real axis. For fermions it is slightly more involved. As described in [3], for these excitations it is often convenient to divide the integration contour into so-called large and small fermion contours.

Finally we have the *matrix part* which encodes the $SU(4)$ R-symmetry structure of the theory. It comes about from the contraction of R-charge indices between the bottom and top pentagons. The matrix part is a coupling independent rational function of (differences of) the rapidities $\{u_i\}$. For generic number of excitations it does not exhibit any obvious factorization, in contrast with the two contributions described thus far. As an example, consider a three-scalar contribution in an NMHV hexagon with two units of R-charge at the bottom and similarly at the top. For such state the matrix part is
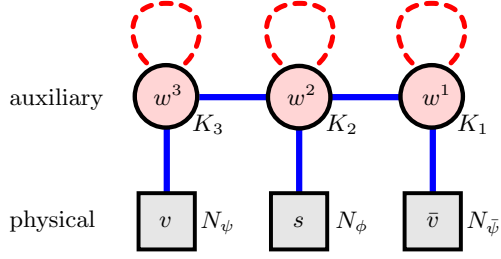
FIG. 1. The pattern of auxiliary rapidities entering the matrix part has a simple group theoretical interpretation. The three sets of rapidities $w^{1,2,3}$ can be identified with the three nodes of the $SU(4)$ Dynkin diagram. The occupation numbers $K_{1,2,3}$ are fixed such that the overall state with N_ϕ scalars, N_ψ fermions and $N_{\bar\psi}$ anti-fermions belongs to the $SU(4)$ representation $\{\mathbf{1}, \bar{\mathbf{4}}, \mathbf{6}, \mathbf{4}, \mathbf{1}\}$ for $r_b = \{0, 1, 2, 3, 4\}$ respectively. The cartoon depicted here is related to (4) by identifying the solid lines with f's and the dashed lines with g's.

given by

$$\Pi_{\text{mat}} = \frac{\left(7 + \sum_i s_i^2 - \sum_{i<j} s_i s_j\right)\left(12 + \sum_i s_i^2 - \sum_{i<j} s_i s_j\right)}{\frac{1}{6i} \prod_{i<j} \left((s_i - s_j)^2 + 1\right)\left((s_i - s_j)^2 + 4\right)} \quad (3)$$

where $\{s_1, s_2, s_3\}$ are the rapidities of the three scalars. The matrix part for states with more scalars or fermions typically lead to gigantic rational functions. Nonetheless, viewed from an appropriate angle, a neat factorization is actually there *even* for the matrix part!

Indeed, it so happens that this matrix part itself admits an integral representation over auxiliary rapidities with a very simple and totally factorized integrand. To describe it one must split the rapidities $\{u_j\}$ into those carried by the scalars, fermions and antifermions, which we denote as $\{s_i\}, \{v_i\}$ and $\{\bar{v}_i\}$, respectively. The remaining rapidities – those of the gluons – do not enter the matrix part since gluons do not carry R-charge. Then

$$\Pi_{\text{mat}} = \frac{1}{K_1! K_2! K_3!} \int \prod_{i=1}^{K_1} \frac{dw_i^1}{2\pi} \int \prod_{i=1}^{K_2} \frac{dw_i^2}{2\pi} \int \prod_{i=1}^{K_3} \frac{dw_i^3}{2\pi}$$
$$\times \frac{g(\mathbf{w}^1)g(\mathbf{w}^2)g(\mathbf{w}^3)}{f(\mathbf{w}^1, \mathbf{w}^2)f(\mathbf{w}^2, \mathbf{w}^3)f(\mathbf{w}^1, \mathbf{v})f(\mathbf{w}^2, \mathbf{s})f(\mathbf{w}^3, \bar{\mathbf{v}})},$$

where all integrals are over the real axis and where we use the short-hand notation

$$g(\mathbf{w}) = \prod_{i<j} g(w_i - w_j), \quad f(\mathbf{w}, \mathbf{v}) = \prod_{i,j} f(w_i - v_j),$$

with $f(x) = x^2 + 1/4$ and $g(x) = x^2(x^2 + 1)$ to describe the fully factorized matrix part integrand, also depicted in figure 1. The numbers of auxiliary roots $K_{1,2,3}$ are given by the solution to

$$\begin{aligned}
N_\psi - 2K_1 + K_2 \phantom{{}- 2K_3} &= \delta_{r_b,3}, \\
N_\phi + K_1 - 2K_2 + K_3 &= \delta_{r_b,2}, \\
N_{\bar\psi} \phantom{{}+ K_1} + K_2 - 2K_3 &= \delta_{r_b,1}.
\end{aligned}$$

All integrals in the matrix part representation can be straightforwardly performed by residues. For an NMHV hexagon with $r_b = 2$, $N_\phi = 3$ and $N_\psi = N_{\bar\psi} = 0$, for instance, we readily reproduce (3) in this way.

This is it. All these ingredients can now be straightforwardly plugged in (1) to yield a fully non-perturbative representation of any six-point scattering amplitudes in planar $\mathcal{N} = 4$ SYM theory at any value of the coupling.

As usual with bootstrap based approaches, our result lies somewhere between a physics conjecture and a mathematical theorem. Any efforts to rigorously establish (1) or to yield further evidence for it would be most valuable.

In the meantime, it is a fascinating problem to investigate the POPE sum in detail and look for physically interesting regimes where it simplifies.

One such limit is weak coupling. At weak coupling almost all integrals can be trivially done. More precisely, all fermion integrals over the so-called small fermion domain can be performed straightforwardly leaving one with the integration over the rapidities of the others excitations – be them large fermions, scalars or gluons. The number of such integrations grows very slowly in perturbation theory. For MHV say, at l loops, we have at most $\sqrt{2l}$ non-trivial integrations to perform! This is extremely economical as compared to any perturbative expansion and is one of the reasons why the POPE approach is so convenient as a boundary data provider for the so-called hexagon bootstrap program [14–18]. Reproducing the full kinematical dependence of the l-loop amplitude, away from the collinear regime, still requires summing over infinite families of flux tube excitations (arising mostly from integrating out any number of small fermions as discussed above). This calls for the development of efficient methods for both performing the OPE integrals and re-summing them, as tackled recently in [19, 20] for the so called gluonic double scaling limit [7].

Another very interesting regime is strong coupling. Equipped with the POPE representation (1) one might want to derive the full strong coupling classical result [21–23], and address the question of computing sub-leading quantum corrections, following analyses of [6, 24–27] for amplitudes and similar problems.

It is an equally fascinating problem to try to access other physically relevant kinematical regimes starting from the OPE sum. One such regime is the Regge limit [28–31] tackled from the POPE lens in [32–34]. It is also very interesting to establish a relation to the Q-bar approach [35, 36] as initiated in [37].

Finally, we look forward to Monte Carlo simulating the flux tube gas (1) on a computer and plot a six gluon amplitude all the way from weak to strong coupling [38].

Rare are the examples of CFT correlators for which the complete OPE series was found. Excitingly, thanks to integrablity, this turned out to be possible here.

ACKNOWLEDGEMENTS

[1] B. Basso, A. Sever and P. Vieira, Phys. Rev. Lett. **111** (2013) 9, 091602.
[2] L. F. Alday, D. Gaiotto, J. Maldacena, A. Sever and P. Vieira, JHEP **1104** (2011) 088.
[3] B. Basso, A. Sever and P. Vieira, JHEP **1401** (2014) 008.
[4] A. V. Belitsky, S. E. Derkachov and A. N. Manashov, Nucl. Phys. B **882**, 303 (2014).
[5] B. Basso, A. Sever and P. Vieira, JHEP **1408**, 085 (2014).
[6] B. Basso, A. Sever and P. Vieira, Phys. Rev. Lett. **113** (2014) 26, 261604.
[7] B. Basso, A. Sever and P. Vieira, JHEP **1409** (2014) 149.
[8] A. V. Belitsky, Nucl. Phys. B **896** (2015) 493.
[9] A. V. Belitsky, Nucl. Phys. B **894** (2015) 108.
[10] B. Basso, J. Caetano, L. Cordova, A. Sever and P. Vieira, JHEP **1508**, 018 (2015).
[11] A. V. Belitsky, Nucl. Phys. B **897**, 346 (2015).
[12] B. Basso, J. Caetano, L. Cordova, A. Sever and P. Vieira, "OPE for all Helicity Amplitudes II. Form Factors and Data analysis."
[13] B. Basso, F. Coronado, A. Sever and P. Vieira, to appear.
[14] L. J. Dixon, J. M. Drummond and J. M. Henn, JHEP **1111** (2011) 023.
[15] L. J. Dixon, J. M. Drummond, M. von Hippel and J. Pennington, JHEP **1312**, 049 (2013).
[16] L. J. Dixon, J. M. Drummond, C. Duhr and J. Pennington, JHEP **1406**, 116 (2014).
[17] L. J. Dixon, J. M. Drummond, C. Duhr, M. von Hippel and J. Pennington, PoS LL **2014** (2014) 077.
[18] L. J. Dixon and M. von Hippel, JHEP **1410** (2014) 65.
[19] J. M. Drummond and G. Papathanasiou, arXiv:1507.08982 [hep-th].
[20] G. Papathanasiou, JHEP **1311** (2013) 150.
[21] L. F. Alday and J. M. Maldacena, JHEP **0706** (2007) 064.
[22] L. F. Alday, D. Gaiotto and J. Maldacena, JHEP **1109** (2011) 032.
[23] L. F. Alday, J. Maldacena, A. Sever and P. Vieira, J. Phys. A **43** (2010) 485401.
[24] J. E. Bourgine, Nucl. Phys. B **880** (2014) 476.
[25] C. Meneghelli and G. Yang, JHEP **1405** (2014) 112.
[26] D. Fioravanti, S. Piscaglia and M. Rossi, Nucl. Phys. B **898** (2015) 301.
[27] J. E. Bourgine and D. Fioravanti, arXiv:1506.01340 [hep-th].
[28] J. Bartels, L. N. Lipatov and A. Sabio Vera, Eur. Phys. J. C **65** (2010) 587.
[29] S. Caron-Huot, JHEP **1505**, 093 (2015).
[30] J. Bartels, J. Kotanski and V. Schomerus, JHEP **1101** (2011) 096.
[31] J. Bartels, J. Kotanski, V. Schomerus and M. Sprenger, arXiv:1311.1512 [hep-th].
[32] B. Basso, S. Caron-Huot and A. Sever, JHEP **1501** (2015) 027.
[33] Y. Hatsuda, JHEP **1410**, 38 (2014).
[34] J. Bartels, L. N. Lipatov and A. Prygarin, arXiv:1104.4709 [hep-th].
[35] M. Bullimore and D. Skinner, arXiv:1112.1056 [hep-th].
[36] S. Caron-Huot and S. He, JHEP **1207** (2012) 174.
[37] A. V. Belitsky, arXiv:1506.02598 [hep-th].
[38] B. Basso, A. Sever and P. Vieira, M. Von Hippel, In progress